

# On Generation of magnetic field in astrophysical bodies

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The generation of magnetic field in astrophysical bodies, e.g., galaxies, stars, planets, is one of the outstanding theoretical problems of physics and astrophysics. The initial magnetic fields of galaxies and stars are weak, and are amplified by the turbulent motion of the plasma. The generated field gets saturated due to nonlinear interactions. The above process is called “dynamo” action. Qualitatively, the magnetic field is amplified by the stretching of the field lines due to turbulent plasma motion. A fraction of kinetic energy of the plasma is spent in increasing the tension of the magnetic field lines, which effectively enhances the magnetic field strength. Current dynamo theories are of two types, kinematic and dynamic. In the kinematic theories, one studies the evolution of magnetic field under a prescribed velocity field. In kinematic  $\alpha$ -dynamo, the averaged nonlinear term  $\langle \mathbf{u} \times \mathbf{b} \rangle$  ( $\mathbf{u}$ ,  $\mathbf{b}$  are velocity and magnetic field fluctuations respectively) is replaced by a constant  $\alpha$  times mean magnetic field  $\mathbf{B}_0$ . This process, which is valid for small magnetic field fluctuations, yields linear equations that can be solved for a given boundary condition and external forcing fields<sup>1–3</sup>. In dynamic theories<sup>4–6</sup>, the modification of velocity field by the magnetic field (back reaction) is taken into account. Using a different approach, here we compute energy transfer rates from velocity field to magnetic field using field-theoretic method. The striking result of our field theoretic calculation is that there is a large energy transfer rate from the large-scale velocity field to the large-scale magnetic field. We claim that the growth of large-scale magnetic energy is primarily due to this transfer. We reached the above conclusion without any linear approximation like that in  $\alpha$ -dynamo.

There is an exchange of energy between various Fourier modes because of nonlinear interactions present in magneto-hydrodynamics (MHD). Since there are two vector fields  $\mathbf{u}$  and  $\mathbf{b}$  in MHD, the energy can be transferred from  $\mathbf{u}$  to  $\mathbf{u}$ ,  $\mathbf{u}$  to  $\mathbf{b}$ , and  $\mathbf{b}$  to  $\mathbf{b}$ . Energy from a parent mode  $\mathbf{u}(\mathbf{k})$  or  $\mathbf{b}(\mathbf{k})$  ( $\mathbf{k}$  represents the wavenumber of Fourier mode) is transferred to two daughter modes with wavenumbers  $\mathbf{p}$  and  $\mathbf{k} - \mathbf{p}$ . The allowed triads in MHD are  $(\mathbf{u}(\mathbf{k}), \mathbf{u}(\mathbf{p}), \mathbf{u}(\mathbf{k} - \mathbf{p}))$  and  $(\mathbf{u}(\mathbf{k}), \mathbf{b}(\mathbf{p}), \mathbf{b}(\mathbf{k} - \mathbf{p}))$ . The net effects of all the energy transfers are constant energy fluxes from large-scale  $u$  to small-scale  $u$  ( $\Pi_{u>}^{u<}$ ), large-scale  $u$  to small-scale  $b$  ( $\Pi_{b>}^{u<}$ ), large-scale  $b$  to small-scale  $u$  ( $\Pi_{u>}^{b<}$ ), large-scale  $b$  to small-scale  $b$  ( $\Pi_{b>}^{b<}$ ), and large-scale  $u$  to large-scale  $b$  ( $\Pi_{b>}^{u>}$ ). The superscript and subscript of  $\Pi$  refer to the source and sink respectively. All these energy fluxes are illustrated in Fig. 1. These energy fluxes are analogous to that of Kolmogorov's flux in fluid turbulence. Note that large-scale velocity modes are forced, as shown in the figure.

Stanišić<sup>7</sup>, Dar et al.<sup>8</sup>, and others have given formulas for computation of the above fluxes. However, Dar et al.'s formalism is the most general, and they have numerically computed all the fluxes of MHD using numerical data of direct numerical simulation<sup>8</sup>. In the present paper, we compute the MHD fluxes in the inertial range using field-theoretic method. Our calculation is up to first order in perturbation.

We give a brief outline of the theoretical calculation (refer to Verma<sup>9</sup> for details). We write down the evolution equation for kinetic energy spectrum ( $\langle |\mathbf{u}(\mathbf{k})|^2 \rangle / 2$ ) and magnetic energy ( $\langle |\mathbf{B}(\mathbf{k})|^2 \rangle / 2$ ). We carry out our analysis for three space dimensions, constant mass density, and zero mean magnetic field. We also assume that  $\langle \mathbf{u} \cdot \mathbf{b} \rangle = 0$ , and that the large-scale velocity modes are forced. There is a net outflow/inflow of energy from a wavenumber sphere (say sphere of radius  $k_0$ ) as discussed in the earlier paragraph. These energy fluxes can be easily calculated using the Fourier modes. To illustrate, the energy flux from the modes inside of the  $u$ -sphere of radius  $k_0$  to the modes outside of the  $b$ -sphere of radius  $k_0$  is given by

$$\Pi_{b>}^{u<}(k_0) = \int_{k'>k_0} \frac{d\mathbf{k}}{(2\pi)^3} \int_{p<k_0} \frac{d\mathbf{p}}{(2\pi)^3} \langle \Im([\mathbf{k} \cdot \mathbf{u}(\mathbf{q})][\mathbf{b}(\mathbf{k}) \cdot \mathbf{u}(\mathbf{p})]) \rangle \quad (1)$$

where  $\Im$  stands for the imaginary part, and  $\mathbf{k} = \mathbf{p} + \mathbf{q}$ . Clearly the sources of energy for the  $b <$  sphere are  $\Pi_{b<}^{u<}$ ,  $\Pi_{b<}^{u>}$ , and  $\Pi_{b<}^{b<}$ . If there is a net flux of energy into large-scale magnetic energy, then magnetic energy at large-scale will grow, or dynamo is active.

We calculate the energy fluxes perturbatively to first order. We assume homogeneity and isotropy for the flow. In the correlation functions we have included kinetic helicity ( $H_K = \langle \mathbf{u} \cdot \boldsymbol{\omega} \rangle / 2$ , where  $\boldsymbol{\omega}$  is the vorticity) and magnetic helicity ( $H_M = \langle \mathbf{a} \cdot \mathbf{b} \rangle / 2$ , where  $\mathbf{a}$  is the vector potential), which are defined using

$$\langle u_i(\mathbf{k}) u_j(\mathbf{k}') \rangle = \left[ P_{ij}(\mathbf{k}) C^{uu}(\mathbf{k}) - i \epsilon_{ijl} k_l \frac{2H_K(k)}{k^2} \right] \delta(\mathbf{k} + \mathbf{k}') \quad (2)$$

$$\langle b_i(\mathbf{k}) b_j(\mathbf{k}') \rangle = \left[ P_{ij}(\mathbf{k}) C^{bb}(\mathbf{k}) - i \epsilon_{ijl} k_l 2H_M(k) \right] \delta(\mathbf{k} + \mathbf{k}') \quad (3)$$

Note that helicities break mirror symmetry.

We focus on the fluxes in the inertial range. Based on recent theoretical<sup>10–14</sup> and numerical<sup>15,16</sup> evidences, we take Kolmogorov's spectrum for the correlation functions in the inertial range, i.e.,

$$C^{uu}(\mathbf{k}) = \frac{K^u}{4\pi} \Pi^{2/3} k^{-11/3}; \quad C^{bb}(\mathbf{k}) = C^{uu}(\mathbf{k})/r_A; \quad (4)$$

$$H_K(\mathbf{k}) = r_K k C^{uu}(\mathbf{k}); \quad H_M(\mathbf{k}) = r_M C^{bb}(\mathbf{k})/k \quad (5)$$

Since both magnetic and kinetic energy spectrum are Kolmogorov-like in the inertial range,  $r_A, r_K$  and  $r_M$  can be treated as constants. We also use turbulent or renormalized viscosity and resistivity in our calculation. These quantities have been recently derived by Verma<sup>13,17</sup>. Using the steady-state condition, we also calculate the energy supply from the large-scale velocity field to the large-scale magnetic field  $\Pi_{b<}^{u<} = \Pi_{b>}^{b<} + \Pi_{u>}^{b<}$ . Since the inertial-range energy spectrum is universal, the above energy flux is independent of the details of large-scale forcing.

We calculate various fluxes in terms of  $r_A, r_K, r_M$  (see Verma<sup>9</sup> for further details). In Fig. 1 we have shown both nonhelical (solid line) and helical (dashed line) contributions for the case ( $r_A = 5000, r_K = 0.1, r_M = -0.1$ ). Large  $r_A$  (here  $r_A = 5000$ ) corresponds to a very weak magnetic field compared to the velocity field, similar to the early phase of galactic and stellar evolution. The choice of  $r_K = 0.1$  and  $r_M = -0.1$  is motivated by the fact that both kinetic and magnetic helicities in most of the astrophysical plasmas are relatively small. The recent simulations of Brandenburg<sup>18</sup> as well as the EDQNM calculation of Pouquet et al.<sup>4</sup> shows that the magnetic helicity is negative for small wavenumbers. Since the small wavenumber contributions dominate the contributions from the large wavenumbers, we have taken  $r_M < 0$ .

The flux ratios shown in Fig. 1 illustrate many important results. They are:

- There is a large energy flux from large-scale velocity field to large-scale magnetic field ( $\Pi_{b<}^{u<}$ ). In addition, there are two other fluxes,  $\Pi_{b<helical}^{b>} + \Pi_{b<helical}^{u>}$ , to the large-scale magnetic field. These fluxes are responsible for the growth of large-scale magnetic field in the initial stage of evolution. Pouquet et al.<sup>4</sup>, Pouquet and Patterson<sup>19</sup>, Brandenburg<sup>18</sup>, and many others generally highlight  $\Pi_{b<helical}^{b>}$  inverse transfer, and do not consider  $\Pi_{b<}^{u<}$ . Recently Brandenburg<sup>18,20</sup> argues that large-scale magnetic energy is sustained by nonlocal inverse cascade from the forcing scale directly to the largest scale of the box. He relates this effect to the  $\alpha$ -effect. In this paper and Verma<sup>9</sup> we have computed the relative magnitudes of all three contributions for generic parameters discussed above, and find that all of them to be comparable, however,  $\Pi_{b<}^{u<}$  is somewhat higher.
- As shown in the figure, the nonhelical component of magnetic energy flux ( $\Pi_{b>}^{b<}$ ) is forward, while the helical component of magnetic energy flux is negative (inverse). The overall magnetic energy flux however is positive.

In our theoretical calculation we have assumed homogeneity and isotropy of the flow, as well as  $\mathbf{B}_0 = \mathbf{0}$ . These assumptions are likely to hold in the early stages of the galactic evolution before large structures appear. To model the early evolution of galaxies, we assume that the large-scales contains kinetic and magnetic energies. During this unsteady period, magnetic energy( $E^b$ ) will be amplified by the nonhelical  $\Pi_{b<}^{u<}$ , and helical components  $\Pi_{b<helical}^{b>}$  and  $\Pi_{b<helical}^{u>}$ , i.e.,

$$\frac{dE^b(t)}{dt} = \Pi_{b<}^{u<} + \Pi_{b<helical}^{b>} + \Pi_{b<helical}^{u>} \quad (6)$$

We assume quasi-steady state for the galactic evolution. Our theoretical calculation performed for large  $r_A$  shows that the energy fluxes of the right-hand-side of Eq. (6) is proportional to  $\Pi E^b/E^u$ , where  $E^u$  is the kinetic energy, and  $\Pi$  is the total energy flux or energy supply rate. Using Kolmogorov's spectrum, we obtain

$$E^b(t) \approx E^b(0) \exp \left( \frac{\sqrt{E^u}}{L(K^u)^{3/2}} t \right) \quad (7)$$

where  $L$  is the large-length of the system. Clearly, the magnetic energy grows exponentially in the early periods, and the time-scale of growth is of the order of  $L/\sqrt{E^u}$ , which is the eddy turnover time. Taking  $L \approx 10^{17} km$  and  $\sqrt{E^u} \approx 10 km/sec$ , we obtain the growth time-scale to be  $10^{16} sec$  or  $3 \times 10^8$  years, which is in the expected range<sup>21</sup>. Hence, we are able to construct a nonlinear and dynamically consistent galactic dynamo based on the energy fluxes. In this model, magnetic energy grows exponentially, and the growth time-scale is reasonable<sup>21</sup>.

To recapitulate, we address the dynamo problem in the light of turbulent energy flux. Our approach is very different from the  $\alpha$ -dynamo picture. Our theoretical calculation, based on perturbative field theory, shows large amount of

energy transfer from large-scale velocity field to large-scale magnetic field. This transfer occurs in both helical and nonhelical MHD. We believe magnetic energy growth is primarily due to this energy transfer. Regarding magnetic energy flux, there is (a) nonhelical forward flux from large scales to small scales, and (b) helical inverse transfer from small scales to large scales. The net magnetic energy transfer, however, is positive.

We have constructed a model of galactic dynamo based on our energy flux results. We find that magnetic energy grows exponentially, and our estimate of its growth time-scale is consistent with the current observational estimates<sup>21</sup>.

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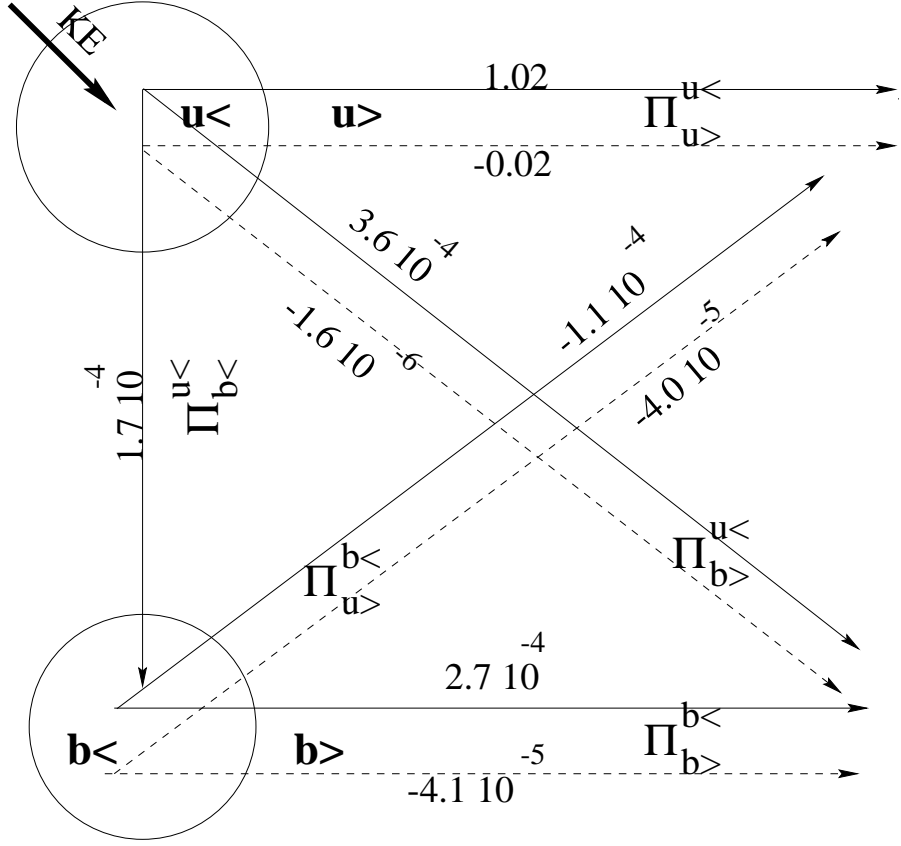


FIG. 1. Various energy fluxes in wavenumber space for parameter ( $r_A = 5000, r_K = 0.1, r_M = -0.1$ ). The illustrated wavenumber spheres contain  $u<$  and  $b<$  modes, while  $u>$  and  $b>$  are modes outside these spheres. The energy fluxes are from large-scale  $u$  to small-scale  $u$  ( $\Pi_{u>}^{u<}$ ), large-scale  $u$  to small-scale  $b$  ( $\Pi_{b>}^{u<}$ ), large-scale  $b$  to small-scale  $u$  ( $\Pi_{u>}^{b<}$ ), large-scale  $b$  to small-scale  $b$  ( $\Pi_{b>}^{b<}$ ), and large-scale  $u$  to large-scale  $b$  ( $\Pi_{b>}^{u>}$ ). The velocity fields at large-scale are forced, and the net input energy is 1 unit. The solid and dashed lines represent the nonhelical and helical contributions respectively.